The mean velocity of slightly buoyant and heavy particles in turbulent flow in a pipe

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SUMMARY

A large number of small spheres of the same size were injected successively into a horizontal pipe conveying water at constant mean velocity, and their times of transit were measured. The mean velocity of the spheres that were either somewhat heavier or lighter than water was less than that of those of neutral density; for those having a terminal velocity in water within $\pm 1\%$ of the mean velocity of the water in the pipe, the discrepancy was only about $0{\cdot}1\%$. The dispersion of the times of transit of the spheres was almost independent of their density. A theory is developed to show how the mean velocity of the spheres depends upon their relative density and size.

1. INTRODUCTION

In a previous paper (Batchelor, Binnie & Phillips 1955) an account was given of experiments in which numerous spheres of almost neutral density were injected in turn into a turbulent stream of water in a horizontal pipe and their passage timed. In this way a method was devised for finding the discharge of water through the pipe and the fluctuation in the times of travel of the spheres. It was shown theoretically that the mean velocity of material particles of the liquid is equal to the discharge velocity U, defined as the discharge, at some cross-section, averaged over a long time and divided by the cross-section. A satisfactory modification was developed to account for the difference between U and the mean velocity $U(\alpha)$ of spheres of small but finite size, this difference being a function of α, the ratio of the diameter of the spheres to that of the pipe. The density of the spheres was such that their terminal velocity in water was within $\pm 1\%$ of the discharge velocity, as determined by a measuring tank at the pipe outlet. This standard of density is not easy to achieve under the conditions of temperature and pressure prevailing in the pipe; and before the method can be used with confidence to determine the discharge velocity through a large pipe-line, it is important to know the consequences when the above standard is slightly relaxed. An attempt to answer this question by means

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of further experiments at the Engineering Laboratory, Cambridge, is described below.

If the spheres are buoyant or heavy, their motion is modified in two ways. Firstly, the difference between the inertia of a sphere and that of an equal volume of fluid results in relative motion when the fluid surrounding the sphere is accelerated. Secondly, the effect of gravity is to modify the probability density of the sphere positions in such a way that for spheres denser than water the probability density is greater towards the bottom of the horizontal pipe than towards the top, and vice versa for spheres lighter than water. It may reasonably be supposed that the effect of gravity on the motion of the spheres is determined by the value of γ, the ratio of their terminal velocity V to the discharge velocity U. The results of the experiments described in § 3 suggest strongly that the inertia effect is negligible when γ is small but that the gravity effect is not; we shall anticipate this result by writing the mean velocity of the spheres as $U(\alpha, \gamma)$.

In the light of experience the apparatus previously used was rebuilt in the manner explained in § 2. The most important improvements were the introduction of an automatic device for injecting the spheres and of better methods of controlling and timing the discharge. More robust spheres were required in the new injection apparatus, and they were made of polythene in place of wax. In the experiments attention was confined to spheres of diameter 0·2 in. in a pipe of diameter 2 in., hence α was fixed at 0·1. The Reynolds number, based on pipe diameter, was about 7×10^4.

The experimental results make it clear that, to achieve the highest accuracy, the 1% standard of terminal velocity should be adhered to. However, the available evidence suggests that little advantage will follow from efforts to improve on this standard. Finally, in § 4, an expression for $U(\alpha, \gamma)$ is derived theoretically, and with the aid of the experimental results tentative predictions are made of its magnitude for values of α other than 0·1.

2. Description of apparatus

The entire pipe-line was rebuilt in new galvanized steel tubing, the mean internal diameter of which in the working section was found by weighing the water required to fill each length. The supply was through a 6-in. main from a large tank about 40 ft. above, in which the level was kept constant within ± 1 in. Near the inlet a stainless-steel orifice-plate was inserted to act as a resistance additional to the wall friction. The experiments were nominally conducted at one velocity only, and all that was required to control the water flow was the complete opening of a valve. Nearby, the spheres were injected by means of the device shown in elevation in figure 1. The long tube **A**, which extended to the centre-line of the pipe, could carry a charge of 100 spheres. The spheres in it were pushed downwards by a small stream of water, supplied at a steady net head of a few inches to the top of the tube and escaping at the bottom into the pipe. The electric motor **B** revolved the disc **C**, which near its periphery carried the vertical pin **D**. Close and parallel to the tube **A** a shaft was mounted to which a small arm was fixed at each end. The upper arm protruded above the disc, so that every 15 sec the shaft was rotated by the pin **D** through 90° and was then returned to its original position by a spring. The bottom arm, which is shown more clearly in the plan and elevation drawn to a larger scale on the left of figure 1, engaged in a slot the lowest sphere in tube **A**; thus, when the shaft rotated, the sphere was moved out into the main stream. Release was assisted by a small vane that deflected the stream downwards into the slot.

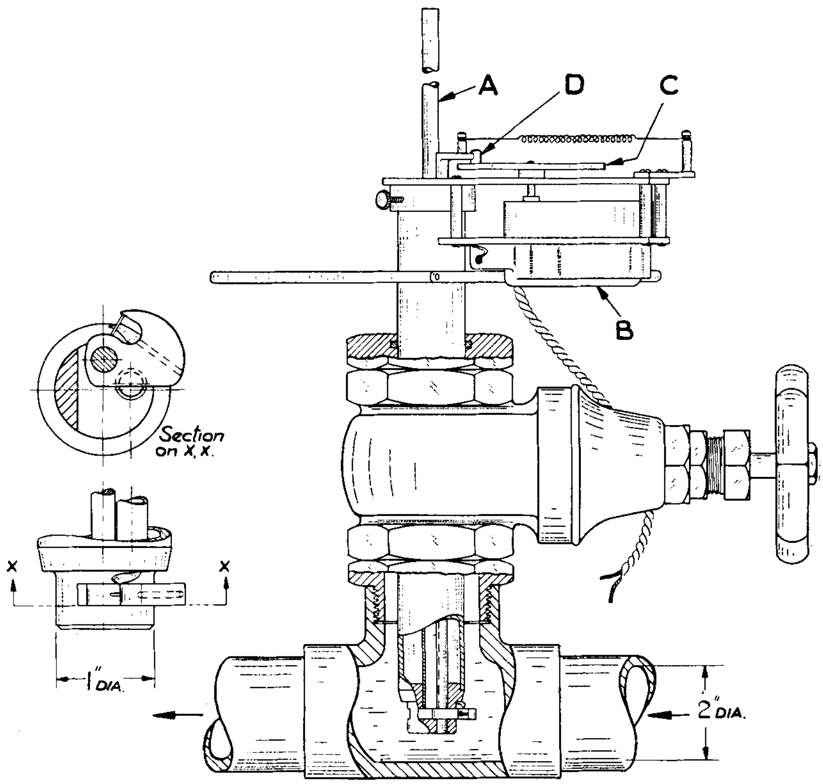

Figure 1. Injection apparatus.

After passing round a bend the water entered on a long, straight and horizontal course. The first 28 ft. served to remove the disturbance due to the bend. The remainder was used for the timing measurements, and in it three lengths of Perspex tube were inserted, fitted with the same arrangement of three photo-cells and two Dekatron counters as was previously employed. Each sphere at the first station A started both counters, and stopped them in turn as it passed stations B and C. The lengths AB and BC were respectively 33·42 and 16·40 ft., and each injection provided two observations directly and one by difference. At exit the water passed into a pool where the spheres were netted and then over a weir into a measuring tank. There the rise was timed at regular intervals with the aid of two vertical probes which started and stopped another Dekatron counter. All the counters were operated by the mains, and long-period errors in the mains frequency did not affect the measured ratio of the velocity of the sphere to the discharge velocity.

Spheres made of the wax previously used were found to be too soft when tried in the new injection device, and this material was replaced by sheet polythene loaded to have a relative density close to unity. At first, the spheres were made in a cold press and by hand moulding after the material had been softened by heat. These processes accidentally produced considerable numbers of spheres in which a small air bubble was trapped. Later, hot injection into a mould was employed usually yielding slightly heavier spheres, and some of these were made heavier still by driving in the point of a pin. The terminal velocity of each sphere was measured in a tube 58 in. long. and the spheres were batched in terms of γ, the ratio of this velocity to the discharge velocity, which was about 5·37 ft./sec. The head in the working section of the pipe was about 3 ft., and the spheres were tested at this pressure in case the presence of the bubbles made the spheres appreciably compressible, but in fact the terminal velocity even of the lightest spheres was found to be independent of the depth of immersion in the tube. The temperature coefficient of expansion of polythene is greater than that of water, and the temperature of the water in the laboratory circulating system was not under control; but its variations were slow, and each batch of spheres was checked at the correct temperature immediately before use. Sufficient spheres were made to permit observations with five batches, for which γ lay within the limits 0·04 to 0·03, 0·02 to 0·01, 0·01 to 0, $-0{\cdot}01$ to $-0{\cdot}02$, $-0{\cdot}02$ to $-0{\cdot}03$, and a further batch at $-0{\cdot}0375$, which was the mean for a mixture of batches $-0{\cdot}03$ to $-0{\cdot}04$ and $-0{\cdot}04$ to $-0{\cdot}05$. Here the negative sign denotes spheres lighter than water. At 15° C the relative densities of spheres for which $\gamma = 0{\cdot}04$ and 0·01 were calculated to be 1·042 and 1·005 respectively.

3. Experimental results

For each value of γ, 200 observations were made of the times of transit of the spheres between the cross-sections AB and AC, measurements of the discharge being taken at the same time at regular intervals. The mean velocity $U(\alpha, \gamma)$ of the spheres was determined from equation (4) of the previous paper, which is

$$U(\alpha,\gamma) \doteqdot x/\overline{T}(x),$$

where $\overline{T}(x)$ is the average time taken for the spheres to travel a length x of the pipe. In this equation there is a proportional error due to dispersion, which can be calculated with the aid of equation (19) of the previous paper. For $\alpha = 0{\cdot}1$ it amounted to about $2a/x$, where a is the pipe radius; and it was ignored because even for the shortest length BC it was only 0·1%.

The measured mean diameters of the lengths AC, AB and BC were respectively 2·005, 2·009 and 1·998 in. (the variation of diameter along each length being unknown). The first of these values was employed in calculating U from the discharge measurements, and the results are shown in figure 2 in the form $\{U(\alpha,\gamma)-U\}/U$ *vs* γ. Now figure 2 of the previous paper shows that for spheres of the same density as water $d\{U(\alpha)/U\}/d\alpha$ is roughly $\frac{1}{2}$ near $\alpha = 0{\cdot}1$. So, if the pipe cross-section is diminished by 1%, thus raising U by the same amount and α by $\frac{1}{2}$%, the increase in $\{U(\alpha)-U\}/U$ is 0·0025. The correction to the observations in AB and AC is therefore small; and it has not been made in figure 2 since a greater precision in the absolute value of U cannot be claimed.

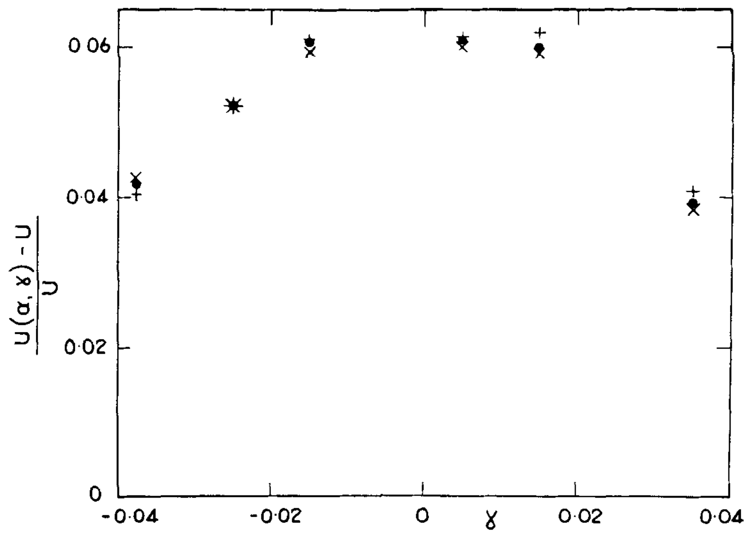

Figure 2. Effect of density upon mean velocity. • AC, × AB, + BC.

Near $\gamma = 0$ the results lie somewhat below the value 0·075 that was found in the earlier work for $\alpha = 0{\cdot}1$. The discrepancy is probably due to the relatively crude method then used for determining the discharge. Even with the more refined method used in the present work, it is difficult to determine U with the accuracy that is readily achieved in the measurement of $U(\alpha,\gamma)$.

The variation of the mean velocity of the spheres with relative density may be due to inertia or to gravity effects (or to a combination of both). The former effect is difficult to analyse, but it is unlikely to be the same for equal and opposite departures of the relative density from 1·0. The gravity effect, on the other hand, is obviously the same for light and heavy spheres whose densities differ from that of water by amounts of the same magnitude. It is also probable that the effect of gravity is to diminish the mean velocity of non-neutral spheres, because the trajectory of lighter or heavier spheres is to a greater extent in regions near the wall at the top or bottom respectively of the pipe, where the velocity of the water is smaller. We do in fact see from figure 2 that the results are almost symmetrical about the line $\gamma = 0$, therefore it was probably the effect of gravity which produced the changes in $U(\alpha, \gamma)$ as γ was varied. The experimental curve is fairly flat over the range $0{\cdot}01 > \gamma > -0{\cdot}01$, but falls rapidly outside these limits. Thus, for $\alpha = 0{\cdot}1$ at any rate, spheres made within these limits will move with very nearly the same mean velocity as spheres of precisely neutral density, the error being no more than $0{\cdot}1\%$.

For a pipe inclined at an angle ϕ to the horizontal, figure 2 might perhaps be used with γ taken as $(V/U)\cos\phi$. However, if ϕ is not a small angle it may be necessary to allow for the gravitational drift of the spheres along the pipe due to the component $V\sin\phi$.

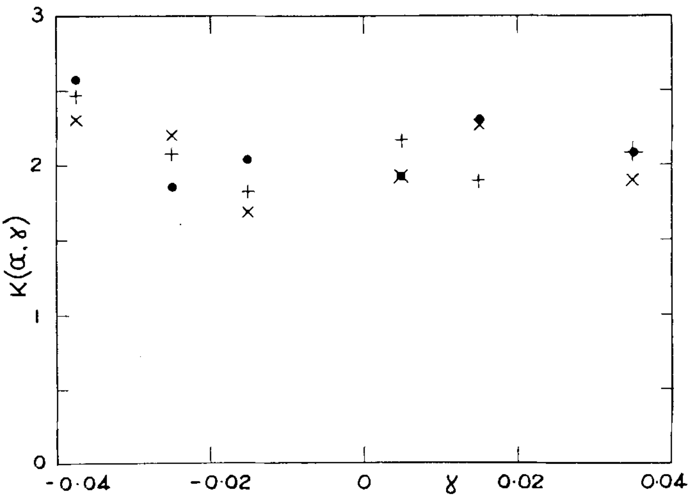

Figure 3. Effect of density upon the dispersion coefficient. • AC, × AB, + BC.

Figure 3 shows the values of the longitudinal dispersion coefficient $K(\alpha,\gamma)$, defined as in equation (19) of the previous paper by

$$K(\alpha,\gamma) = \frac{x}{2a}\frac{U}{u_\tau}\frac{\overline{[T(x)-\overline{T}(x)]^2}}{\overline{T}^2(x)},$$

where u_τ is the friction velocity, and $T(x)$ is the time of travel of a single sphere. Unlike $\{U(\alpha,\gamma)-U\}/U$, the value of K is not sensitive to small changes in γ, and the agreement at $\gamma = 0$ with the earlier work is satisfactory. But the accuracy of these measurements involving the variance of the travel times is not high. The standard deviation of the measured value of K is given by elementary statistical theory as a fraction $n^{-1/2}$ of the true value, where n is the number of observations. Here $n = 200$, so that the expected standard deviation is about 14%, which is in accordance with the scatter found. However, the results do seem to have a trace of a minimum near $\gamma = 0$. Since the longitudinal dispersion is largely a result of the intermittent excursions of the spheres into the region of lower velocity near the wall, an increase in $K(\alpha, \gamma)$ when γ is different from zero is consistent with the above interpretation of the results shown in figure 2.

4. Theoretical discussion

There remains the question of accounting for these observations by an approximate theory so that predictions can be made concerning the behaviour of $\{U(\alpha, \gamma) - U\}/U$ for sphere sizes other than that ($\alpha = 0{\cdot}1$) used in these experiments. If the spheres are not of neutral density, that is, if $\gamma \neq 0$, the probability density p of the sphere distribution (defined so that the probability of finding the centre of the sphere in an element dA of the cross-section is $p\,dA$) is not uniform over the cross-section. The observations show that inertia effects can be neglected as a first approximation when γ is small, and it can be supposed that as a result of gravity the probability density p is a function of the vertical position coordinate z only, that is, that the lines of equal probability density are horizontal. Then, if V represents the terminal velocity of the spheres (a positive sign indicating motion upwards) and if conditions are statistically stationary in time, the probable number of spheres rising across unit horizontal area per unit time as a result of the action of gravity alone is $p(z)V$, which must be balanced by the downward flux due to turbulent diffusion, $u_\tau a\zeta\, dp(z)/dz$, where u_τ is the friction velocity and ζ is a dimensionless turbulent diffusivity for transport across horizontal planes. In general, ζ is also a function of position in the pipe, and depends upon the relative intensities of the turbulent fluctuations in the vertical direction at different points in the cross-section. However, if the sphere size is not so small that its motion is affected by excursions into the viscous sub-layer, it is confined to the central portion of the flow where the turbulent intensities do not vary by a factor of more than about $1{\cdot}5$. A further approximation, consistent with taking $p = p(z)$, is to neglect the variation in ζ with position, and to consider it as a mean vertical diffusivity for the turbulent flow in a pipe. We then have

$$\frac{dp(z)}{dz} = \frac{V}{u_\tau a\zeta} p(z). \tag{1}$$

so that

$$p(z') = p_0 \exp(\gamma U z'/u_\tau \zeta), \tag{2}$$

where $z' = z/a$. The constant of integration p_0 is determined by the condition that $p(z')$ integrated over that part of the cross-section accessible to the centre of the sphere, that is, a circle of radius $(1-\alpha)a$, must equal unity. When γ is small, the exponential can be expanded as a power series, and the integration is found to give the result

$$p_0 = \{\pi a^2(1-\alpha)^2\}^{-1}\{1 - \tfrac{1}{8}(\gamma U/u_\tau \zeta)^2(1-\alpha)^2 + O(\gamma^4)\}. \tag{3}$$

Let us now make the further assumption that the mean velocity of spheres whose centres are at a radial position r in the pipe is equal to the mean velocity $u(r)$ of the fluid at radius r, which was shown in figure 2 of the previous paper to be a good approximation for $\alpha < 0{\cdot}15$. Taking $z = r\cos\theta$ and $r' = r/a = z'/\cos\theta$, we see that the mean velocity of a sphere down the pipe is given by

$$U(\alpha, \gamma) = a^2 \int_0^{2\pi} \int_0^{1-\alpha} p(z')u(r')r'\, dr'd\theta. \tag{4}$$

Substituting from (2) and (3) into (4) and integrating with respect to θ, we have, neglecting terms of order γ^4,

$$U(\alpha,\gamma) = U(\alpha) - \left(\frac{\gamma U}{(1-\alpha)u_\tau \zeta}\right)^2 \left\{\tfrac{1}{4}(1-\alpha)^2 \int_0^{1-\alpha} ru(r)\, dr - \tfrac{1}{2}\int_0^{1-\alpha} r^3 u(r)\, dr\right\}, \tag{5}$$

where the accents on r' are suppressed and

$$U(\alpha) = U(\alpha, \gamma = 0) = \{\pi(1-\alpha)^2\}^{-1} \int_0^{2\pi}\int_0^{1-\alpha} u(r)r\, drd\theta.$$

Taylor (1954) shows that $u(r)$ is of the form

$$u(r) = U + u_\tau\{4{\cdot}25 - f(r)\}, \tag{6}$$

where $f(r)$ is a universal function, and that for Reynolds numbers of order 10^5, as in our experiments, $U/u_\tau \doteqdot 22$. Hence

$$u(r) = U\{1{\cdot}193 - 0{\cdot}0455 f(r)\}. \tag{7}$$

Equation (5) can then be expressed as

$$\frac{U(\alpha,\gamma) - U(\alpha)}{U} = -\frac{484}{(1-\alpha)^2}\frac{\gamma^2}{\zeta^2}\{\tfrac{1}{4}(1-\alpha)^2 K_1(\alpha) - \tfrac{1}{2}K_3(\alpha)\}, \tag{8}$$

where

$$\left.\begin{aligned} K_1(\alpha) &= \frac{1}{U}\int_0^{1-\alpha} ru(r)\, dr = 0{\cdot}597(1-\alpha)^2 - 0{\cdot}0455 \int_0^{1-\alpha} rf(r)\, dr, \\ K_3(\alpha) &= \frac{1}{U}\int_0^{1-\alpha} r^3 u(r)\, dr = 0{\cdot}299(1-\alpha)^4 - 0{\cdot}0455 \int_0^{1-\alpha} r^3 f(r)\, dr. \end{aligned}\right\} \tag{9}$$

α	$K_1(\alpha)$	$K_3(\alpha)$
0·02	0·4899	0·2179
0·04	0·4749	0·2046
0·06	0·4599	0·1913
0·08	0·4460	0·1781
0·10	0·4341	0·1650

Table 1. Values of $K_1(\alpha)$ and $K_3(\alpha)$.

The integrals on the right of the equations (9) were calculated from the figures given in Taylor's table 1, and the quantities $K_1(\alpha)$ and $K_3(\alpha)$ are given in table 1 for various small values of α. For $\alpha = 0{\cdot}1$ it is then found from (8) that

$$\frac{U(\alpha,\gamma) - U(\alpha)}{U} = -3{\cdot}30\frac{\gamma^2}{\zeta^2}. \tag{10}$$

The experiments with $\alpha = 0{\cdot}1$ described in §3 enable an estimate to be made of the dimensionless diffusivity ζ. The ratio $\{U(\alpha,\gamma)-U\}/U$ for $\alpha = 0{\cdot}1$ and $\gamma = 0$ was judged to be 0·0615 from figure 2; and by subtraction, values of $\{U(\alpha,\gamma)-U(\alpha)\}/U$ were obtained. The parameter ζ was then chosen so that the parabolic form (10) fitted these observations most closely, and the value required was found to be 0·46. The mean vertical diffusivity is therefore $0{\cdot}46au_\tau$. It is interesting to notice that the radial eddy diffusivity in turbulent pipe flow found in another way is of the same order as this, some experiments of Schwarz & Hoelscher (1956) with water vapour in air indicating a mean value of approximately $0{\cdot}55au_\tau$ in the central region of the pipe.

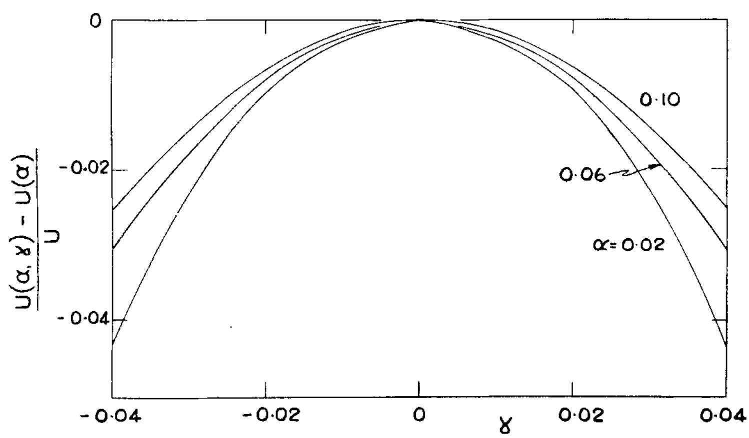

Figure 4. Theoretical effect of density upon mean velocity.

With the value of ζ determined above, the form of $\{U(\alpha,\gamma)-U(\alpha)\}/U$ for small values of γ and for $\alpha = 0{\cdot}06$ and $0{\cdot}02$ was found from equation (8), and the results are shown in figure 4. For a given value of γ, the difference between the mean velocity of the spheres down the pipe and the mean velocity of those for which $\gamma = 0$ increases with decreasing diameter ratio α. The physical reason for this greater effect of gravity on smaller spheres is that small spheres are free to move closer to the walls where the mean velocity of the fluid is less and is changing rapidly; the net effect, then, of the increased probability of finding (say) a heavy sphere near the lower wall and the decreased probability of finding it near the wall in the upper part of the pipe is that, for a given value of γ, $\{U(\alpha,\gamma)-U(\alpha)\}/U$ increases in absolute value as α decreases. These curves indicate that, provided γ is held within the limits $\pm 0{\cdot}01$, the difference between the mean velocity of such spheres and of those for which γ is accurately zero is likely to be negligible, as has already been established experimentally for $\alpha = 0{\cdot}1$.

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References

Batchelor, G. K., Binnie, A. M., & Phillips, O. M. 1955 *Proc. Phys. Soc.* B, **68**, 1095.

Schwarz, W. H., & Hoelscher, H. E. 1956 *J. Amer. Inst. Chem. Engrs.* **2**, 101.

Taylor, G. I. 1954 *Proc. Roy. Soc.* A, **223**, 446.